

ST740 Exam

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1 Introduction

As we begin to feel the ramifications of our industrial haphazardness, the forecasting of climate under previously unrecorded conditions grows more important. One of our greatest assets in this venture is data-driven forecasting. Unfortunately for us, we have been in the habit of collecting data for a relatively small period in the 4.6 billion years since Earth's atmosphere formed.

One remedy to this issue is to take proxy variables of which we may have longer records of and relate them to the climatic process of interest. In Cahill, et.al (2023)¹, they construct a Bayesian time-series model to reconstruct historic RFI based on six independent proxy variables. In this paper, we will attempt to replicate a version of their model.

2 Model Overview

2.1 Model Formulation

Our data consists of proxy variables X_{jt} and response Y_t with $t = 1, \dots, n$, $j = 1, \dots, 6$. Each of the proxies are centered and scaled to have variance 1. Additionally, we centered and scaled the response.

For this problem we model each proxy and the RFI itself as autoregressive processes. Further, we let the means of the proxies be linearly dependent upon the RFI. The model is defined as follows:

$$Y_t|Y_{t-1} \sim N(\mu_0 + \phi_0(Y_{t-1} - \mu_0), (1 - \phi_0^2)\sigma_0^2)$$

$$X_{jt}|X_{j,t-1} \sim N(\mu_{jt} + \phi_j(X_{j,t-1} - \mu_{jt}), (1 - \phi_j^2)\sigma_j^2)$$

$$E[X_{jt}] = \mu_{jt} = \alpha_j + \beta_j Y_t$$

Our model parameter is: $\theta = (\mu_0, \sigma_0, \phi_0, \sigma, \beta, \alpha, \phi)$ where $\sigma = (\sigma_1, \dots, \sigma_6)$, $\beta = (\beta_1, \dots, \beta_6)$, $\alpha = (\alpha_1, \dots, \alpha_6)$, $\phi = (\phi_1, \dots, \phi_6)$. Additionally, we say that $Y_1 \sim N(\mu_0, \sigma_0^2)$ and $X_{j1} \sim N(\mu_j, \sigma_j^2)$. Finally we put the following priors on our parameters: $\beta_j, \mu_0 \sim N(0, 100^2)$, $\sigma_0^2, \sigma_j^2 \sim IG(0.1, 0.1)$, and $\phi_0, \phi_j \sim Unif(0, 1)$.

2.2 Full Conditional Posteriors

First, in order to see the relationship between RFI and the proxies, we let $\phi_0 = \dots = \phi_6 = 0$, removing the autocorrelation across time. Our goal is to reconstruct the RFI, so let us look at the posterior of $Y_t | X_{1t}, \dots, X_{6t}$ (derivations are given in Appendix A).

$$Y_t | X_{1t}, \dots, X_{6t} \sim N \left(\left(\frac{\mu_0}{\sigma_0^2} + \sum_{j=1}^6 \frac{\beta_j (x_{jt} - \alpha_j)}{\sigma_j^2} \right) \left(\frac{1}{\sigma_0^2} + \sum_{j=1}^6 \frac{\beta_j}{\sigma_j^2} \right)^{-1}, \left(\frac{1}{\sigma_0^2} + \sum_{j=1}^6 \frac{\beta_j}{\sigma_j^2} \right)^{-1} \right)$$

This indicates that Y_t is directly dependent upon X_{jt} through its mean structure. There is also a weighting of each proxy variable based on their individual variances when calculating the conditional mean. An additional point of interest is that the conditional variance contains β_j . Specifically, it contains β_j in a way such that larger values (i.e. stronger correlations between response and proxy) will decrease the overall variance of the conditional distribution.

Now, we want to restore the autoregressive features of the model and derive the full conditional posterior of Y_t . First, we note that the autoregressive processes given are first-order, and therefore, the joint distribution of Y can be re-expressed as

$$p(Y_1, \dots, Y_n) = \prod_{t=2}^n p(Y_t | Y_{t-1}) \cdot p(Y_1)$$

Taking this, we derive the following full conditional posterior for Y_t with further details available in the Appendix.

$$Y_t | \dots \sim N(MV^{-1}, V^{-1})$$

where

$$M = \sum_{j=1}^6 \frac{(x_{jt} - \phi_j x_{j,t-1} - (1 - \phi_j) \alpha_j) (1 - \phi_j) \beta_j}{(1 - \phi_j^2) \sigma_j^2} + \frac{y_{t+1} - (1 - \phi_0) \mu_0 \phi_0 + (\mu_0 + \phi_0 (y_{t-1} - \mu_0))}{(1 - \phi_0^2) \sigma_0^2}$$

$$V = \sum_{j=1}^6 \frac{(1 - \phi_j)^2 \beta_j^2}{(1 - \phi_j^2) \sigma_j^2} + \frac{(1 + \phi_0^2)}{(1 - \phi_0^2) \sigma_0^2}$$

Similarly, we derived the following conditional posteriors for X_{jt} and β_j .

$$X_t | \dots \sim N(MV^{-1}, V^{-1})$$

where

$$M = \frac{\mu_{jt} + \phi_j(x_{j,t-1} - \mu_{jt}) + (x_{j,t+1} - (1 - \phi_j)\mu_{j,t+1})\phi_j}{(1 - \phi_j^2)\sigma_j^2}$$

$$V = \frac{(1 + \phi_j^2)}{(1 - \phi_j^2)\sigma_j^2}$$

and

$$\beta_j | \dots \sim N(MV^{-1}, V^{-1})$$

where

$$M = \sum_{t=2}^n \frac{(x_{jt} - \phi_j x_{j,t-1} - (1 - \phi_j)\alpha_j)(1 - \phi_j)y_t}{(1 - \phi_j^2)\sigma_j^2} + \frac{(x_{j1} - \alpha_j)y_1}{\sigma_j^2}$$

$$V = \sum_{t=2}^n \frac{(1 - \phi_j)^2 y_t^2}{(1 - \phi_j^2)} + \frac{y_1^2}{\sigma_j^2} + \frac{1}{100^2}$$

3 Computation

Our goal is to obtain samples from the joint posterior of all parameters and missing values of Y and X_j . For this, we constructed an MCMC sampler that uses Gibbs sampling for all conditional posteriors with the exception of the ϕ_0, \dots, ϕ_6 posteriors, which required a Metropolis-Hastings algorithm with a Beta proposal distribution.

When attempting to sample from the joint posterior including both α_j and β_j , we ran into convergence/identifiability issues. However, this did not occur when we fixed one and sampled from the other. For this reason, and our interest in β_j , we fixed each $\alpha_j = 0$. In order to show this still yielded valid inference, we fit the same model on synthetic data with α_j fixed. Every run of the simulation gave a 95% credible interval for Y that had $> 85\%$ coverage, which we consider to be satisfactory. Inference on β_j was similarly still acceptable. Code for the synthetic data generation can be found in Appendix B.

Each of the other full conditionals used can be found in Appendix A and code can be found in Appendix B. Figure 1 contains the trace plots of all β_j and ϕ_j posterior draws. Visually, it appears that these two sets of variables both converged to their posteriors. To confirm this, we perform a Geweke diagnostic test on all MCMC chains. The largest absolute z-score returned was ≈ 1.8 , which is less than 2, while the majority were under 1. Thus, we are satisfied that all of our MCMC chains converged.

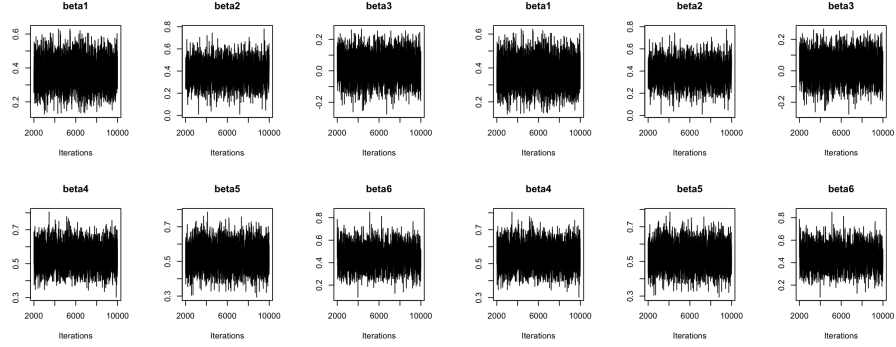


Figure 1: β_j (left) and ϕ_j (right) trace plots

It is not easy to see in these figures, but both of those parameter sets had higher autocorrelations than we would like. For that reason, we ran 20,000 iterations and discarded the first 2,000. To check further, we calculated the effective sample size. Our lowest ESS came from the posterior of ϕ_0 with 1,300. We consider this satisfactory, as all other problematic variables had ESS ranging from between 2,000 and 6,0000. Therefore, we can conclude we have adequately sampled from the joint posterior.

4 Analysis

First, we conduct tests of the β_j coefficients to determine if there is a significant relationship between RFI and each of the proxy variables. Below are the 95% credible intervals of each.

Quantile:	0.025	0.5	0.975
β_1	0.228	0.372	0.524
β_2	0.198	0.380	0.562
β_3	-0.125	0.021	0.167
β_4	0.422	0.540	0.667
β_5	0.407	0.529	0.656
β_6	0.274	0.450	0.630

As we can see, β_3 has 0 in its 95% credible interval so we conclude that the third proxy is not significantly related. Otherwise, we are satisfied that the remaining proxies are significant.

The first graph on the left of figure 2 depicts the RFI reconstructed with our model from 1850-1950. The shaded grey area is a 95% credible interval around our predictions. As we can see, our model seems to pair up well with the actual RFI data we have for the second half of that period. In particular, we can see our predicted trend rising from a lower position up to meet the peak that occurs around 1890.

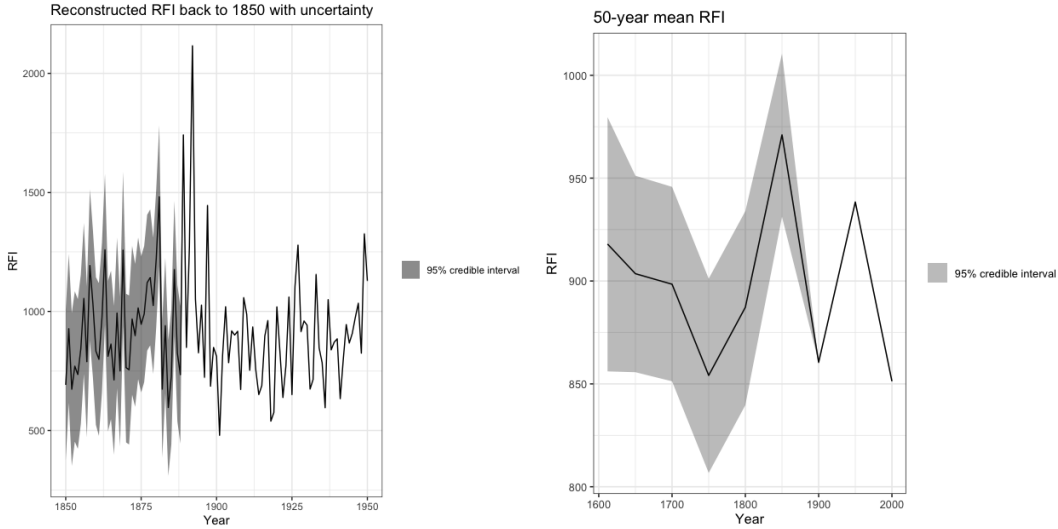


Figure 2

This indicates a reasonable imputation, at least visually.

Finally, in the second graph on the right of Figure 2, we took 50-year means of our imputed/actual RFI and looked at the trend. Specifically, we are interested in whether any of the previous time periods has a lower mean RFI than the most recent period. Using our credible intervals, we find that any of our imputed ranges have at lowest a 0.025 quantile of ≈ 860 . However, this is still higher than the actual mean of ≈ 850 in our recent period. Thus, we conclude that since 1850, the average RFI has never been lower than the period from 2000-2017, which indicates a historically dry period for Brisbane.

5 Conclusion

Ultimately, we were able to achieve our goal of imputing historical RFI based on its relationship to several significant proxy variables. However, we can still identify a few limitations. Firstly, in our analysis we treated each of the proxy variables as independent from one another. We know that environmental processes are often deeply and complexly interlinked; therefore, in a future analysis it would be interesting to construct a more cohesive dependence structure, such that all proxies can borrow information from one another when we have to impute those missing values.

A second extension of this analysis would be to consider multiple locations and implement a spatial correlation structure on top of the time series. Once again, environmental processes are highly correlated across space, so we may be able to borrow even more information across other sites and more accurately impute historical RFI values.

6 References

1. Cahill, N., Croke, J., Campbell, M., Hughes, K., Vitkovsky, J., Kilgallen, J. E., & Parnell, A. (2023). A Bayesian time series model for reconstructing hydroclimate from multiple proxies. *Environmetrics*, 34(4), e2786. <https://doi.org/10.1002/env.2786>

7 Appendix A: Derivations

7.1 No autocorrelation proxy relationships

Letting $\phi_0 = \dots = \phi_6 = 0$, we get the following distribution for a given Y_t :

$$\begin{aligned}
 p(Y_t|X_{1t}, \dots, X_{6t}) &\propto p(X_{1t}, \dots, X_{6t}|Y_t)p(Y_t) \\
 &\propto \prod_{j=1}^6 p(X_{jt}|Y_t) \cdot p(Y_t) \\
 &\propto \exp \left\{ \sum_{j=1}^6 -\frac{1}{2\sigma_j^2} (x_{jt} - \alpha_j - \beta_j y_t)^2 - \frac{1}{2\sigma_0^2} (y_t - \mu_0)^2 \right\} \\
 &\propto \exp \left\{ -\frac{1}{2} \left[-2 \left(\frac{\mu_0}{\sigma_0^2} + \sum_{j=1}^6 \frac{\beta_j (x_{jt} - \alpha_j)}{\sigma_j^2} \right) y_t + \left(\frac{1}{\sigma_0^2} + \sum_{j=1}^6 \frac{\beta_j}{\sigma_j^2} \right) y_t^2 \right] \right\} \\
 \implies Y_t|X_{1t}, \dots, X_{6t} &\sim N \left(\left(\frac{\mu_0}{\sigma_0^2} + \sum_{j=1}^6 \frac{\beta_j (x_{jt} - \alpha_j)}{\sigma_j^2} \right) \left(\frac{1}{\sigma_0^2} + \sum_{j=1}^6 \frac{\beta_j}{\sigma_j^2} \right)^{-1}, \left(\frac{1}{\sigma_0^2} + \sum_{j=1}^6 \frac{\beta_j}{\sigma_j^2} \right)^{-1} \right)
 \end{aligned}$$

7.2 Y_t full conditional

$$\begin{aligned}
 p(Y_t|\dots) &\propto \left[\prod_{j=1}^6 p(X_{jt}|Y_t) \right] p(Y_{t+1}|Y_t)p(Y_t|Y_{t-1}) \\
 p(Y_{t+1}|Y_t)p(Y_t|Y_{t-1}) &\propto \exp \left\{ -\frac{1}{2(1-\phi_0^2)\sigma_0^2} [y_{t+1} - \mu_0 - \phi_0(y_t - \mu_0)]^2 \right\} \\
 &\quad \cdot \exp \left\{ -\frac{1}{2(1-\phi_0^2)\sigma_0^2} [y_t - \mu_0 - \phi_0(y_{t-1} - \mu_0)]^2 \right\} \\
 &\propto \exp \left\{ -\frac{1}{2(1-\phi_0^2)\sigma_0^2} [(y_{t+1} - (1-\phi_0)\mu_0) - \phi_0 y_t]^2 \right\} \\
 &\quad \cdot \exp \left\{ -\frac{1}{2(1-\phi_0^2)\sigma_0^2} [y_t - (\mu_0 + \phi_0(y_{t-1} - \mu_0))]^2 \right\}
 \end{aligned}$$

$$\begin{aligned}
& \propto \exp \left\{ -\frac{1}{2(1-\phi_0^2)\sigma_0^2} [-2(y_{t+1} - (1-\phi_0)\mu_0)\phi_0 y_t + \phi_0^2 y_t^2 \right. \\
& \quad \left. -2(\mu_0 + \phi_0(y_{t-1} - \mu_0))y_t + y_t^2] \right\} \\
& \propto \exp \left\{ -\frac{1}{2(1-\phi_0^2)\sigma_0^2} [-2\left((y_{t+1} - (1-\phi_0)\mu_0)\phi_0 + (\mu_0 + \phi_0(y_{t-1} - \mu_0)) \right) y_t + (1+\phi_0^2)y_t^2] \right\} \\
& \implies Y_t | Y_{t-1}, Y_{t+1} \sim N \left(\frac{y_{t+1} - (1-\phi_0)\mu_0\phi_0 + (\mu_0 + \phi_0(y_{t-1} - \mu_0))}{1+\phi_0^2}, \frac{(1-\phi_0^2)\sigma_0^2}{1+\phi_0^2} \right)
\end{aligned}$$

Now, to add in proxy dependence.

$$\begin{aligned}
p(X_{jt}|Y_t) & \propto \exp \left\{ \frac{1}{2(1-\phi_j^2)\sigma_j^2} [x_{jt} - \alpha_j - \beta_j y_t - \phi_j(x_{j,t-1} - \alpha_j - \beta_j y_t)]^2 \right\} \\
& \propto \exp \left\{ \frac{1}{2(1-\phi_j^2)\sigma_j^2} [-2(x_{jt} - \phi_j x_{j,t-1} - (1-\phi_j)\alpha_j)(1-\phi_j)\beta_j y_t + (1-\phi_j)^2 \beta_j^2 y_t^2] \right\}
\end{aligned}$$

Thus,

$$\begin{aligned}
p(Y_t | \dots) & \propto \left[\prod_{j=1}^6 p(X_{jt}|Y_t) \right] p(Y_{t+1}|Y_t) p(Y_t|Y_{t-1}) \\
& \propto \exp \left\{ -\sum_{j=1}^6 \frac{1}{2(1-\phi_j^2)\sigma_j^2} [-2(x_{jt} - \phi_j x_{j,t-1} - (1-\phi_j)\alpha_j)(1-\phi_j)\beta_j y_t + (1-\phi_j)^2 \beta_j^2 y_t^2] \right. \\
& \quad \left. -\frac{1}{2(1-\phi_0^2)\sigma_0^2} [-2((y_{t+1} - (1-\phi_0)\mu_0)\phi_0 + (\mu_0 + \phi_0(y_{t-1} - \mu_0)))y_t + (1+\phi_0^2)y_t^2] \right\} \\
& \propto \exp \left\{ -\frac{1}{2} \left[-2 \left(\sum_{j=1}^6 \frac{(x_{jt} - \phi_j x_{j,t-1} - (1-\phi_j)\alpha_j)(1-\phi_j)\beta_j}{(1-\phi_j^2)\sigma_j^2} \right. \right. \right. \\
& \quad \left. \left. + \frac{y_{t+1} - (1-\phi_0)\mu_0\phi_0 + (\mu_0 + \phi_0(y_{t-1} - \mu_0))}{(1-\phi_0^2)\sigma_0^2} \right) y_t + \left(\sum_{j=1}^6 \frac{(1-\phi_j)^2 \beta_j^2}{(1-\phi_j^2)\sigma_j^2} + \frac{(1+\phi_0^2)}{(1-\phi_0^2)\sigma_0^2} \right) y_t^2 \right] \right\}
\end{aligned}$$

Now let

$$\begin{aligned}
V & = \sum_{j=1}^6 \frac{(1-\phi_j)^2 \beta_j^2}{(1-\phi_j^2)\sigma_j^2} + \frac{(1+\phi_0^2)}{(1-\phi_0^2)\sigma_0^2} \\
M & = \sum_{j=1}^6 \frac{(x_{jt} - \phi_j x_{j,t-1} - (1-\phi_j)\alpha_j)(1-\phi_j)\beta_j}{(1-\phi_j^2)\sigma_j^2} + \frac{y_{t+1} - (1-\phi_0)\mu_0\phi_0 + (\mu_0 + \phi_0(y_{t-1} - \mu_0))}{(1-\phi_0^2)\sigma_0^2} \\
& \implies Y_t | \dots \sim N(MV^{-1}, V^{-1})
\end{aligned}$$

7.3 X_t full conditional

$$\begin{aligned}
& p(X_{jt} | \dots) \propto p(X_{j(t+1)} | X_{jt}, Y_{t+1}) p(X_{jt} | X_{j(t-1)}, Y_t) \\
& \propto \exp \left\{ -\frac{1}{2(1-\phi_j^2)\sigma_j^2} [x_{jt} - (\mu_{jt} + \phi_j(x_{j,t-1} - \mu_{jt}))]^2 \right\} \\
& \cdot \exp \left\{ -\frac{1}{2(1-\phi_j^2)\sigma_j^2} [x_{j,t+1} - (\mu_{j,t+1} + \phi_j(x_{jt} - \mu_{j,t+1}))]^2 \right\} \\
& \propto \exp \left\{ -\frac{1}{2(1-\phi_j^2)\sigma_j^2} [x_{jt} - (\mu_{jt} + \phi_j(x_{j,t-1} - \mu_{jt}))]^2 \right\} \\
& \cdot \exp \left\{ -\frac{1}{2(1-\phi_j^2)\sigma_j^2} [(x_{j,t+1} - (1-\phi_j)\mu_{j,t+1}) - \phi_j x_{jt}]^2 \right\} \\
& \propto \exp \left\{ -\frac{1}{2(1-\phi_j^2)\sigma_j^2} [-2(\mu_{jt} + \phi_j(x_{j,t-1} - \mu_{jt}))x_{jt} + x_{jt}^2 \right. \\
& \quad \left. -2((x_{j,t+1} - (1-\phi_j)\mu_{j,t+1})\phi_j)x_{jt} + \phi_j^2 x_{jt}^2] \right\} \\
& \propto \exp \left\{ -\frac{1}{2(1-\phi_j^2)\sigma_j^2} [-2(\mu_{jt} + \phi_j(x_{j,t-1} - \mu_{jt})) + (x_{j,t+1} - (1-\phi_j)\mu_{j,t+1})\phi_j]x_{jt} \right. \\
& \quad \left. + (1 + \phi_j^2)x_{jt}^2 \right\} \\
& \implies X_t | \dots \sim N(MV^{-1}, V^{-1})
\end{aligned}$$

where

$$\begin{aligned}
M &= \frac{\mu_{jt} + \phi_j(x_{j,t-1} - \mu_{jt}) + (x_{j,t+1} - (1-\phi_j)\mu_{j,t+1})\phi_j}{(1-\phi_j^2)\sigma_j^2} \\
V &= \frac{(1 + \phi_j^2)}{(1-\phi_j^2)\sigma_j^2}
\end{aligned}$$

7.4 β_j full conditional

$$\begin{aligned}
& p(\beta_j | \dots) \propto p(\mathbf{X}_j | \beta_j) \pi(\beta_j) \\
& \propto \prod_{t=2}^n p(X_{jt} | X_{j,t-1}, \beta_j) \cdot p(X_{j1} | \beta_j) \pi(\beta_j) \\
& \propto \exp \left\{ \sum_{t=2}^n -\frac{1}{2(1-\phi_j^2)\sigma_j^2} [x_{jt} - (\mu_{jt} + \phi_j(x_{j,t-1} - \mu_{jt}))]^2 \right\} \exp \left\{ -\frac{1}{2\sigma_j^2} [x_{j1} - \mu_{j1}]^2 \right\} \\
& \quad \cdot \exp \left\{ -\frac{1}{2(100^2)} \beta_j^2 \right\}
\end{aligned}$$

$$\begin{aligned}
& \propto \exp \left\{ \sum_{t=2}^n -\frac{1}{2(1-\phi_j^2)\sigma_j^2} [x_{jt} - \phi_j x_{j,t-1} - (1-\phi_j)\alpha_j - (1-\phi_j)\beta_j y_t]^2 \right\} \\
& \quad \cdot \exp \left\{ -\frac{1}{2\sigma_j^2} [x_{j1} - \alpha_j - \beta_j y_1]^2 - \frac{1}{2(100^2)\beta_j^2} \right\} \\
& \propto \exp \left\{ \sum_{t=2}^n -\frac{1}{2(1-\phi_j^2)\sigma_j^2} [-2(x_{jt} - \phi_j x_{j,t-1} - (1-\phi_j)\alpha_j)(1-\phi_j)y_t \beta_j + (1-\phi_j)^2 y_t^2 \beta_j^2] \right. \\
& \quad \left. - \frac{1}{2\sigma_j^2} [-2(x_{j1} - \alpha_j)y_1 \beta_j + y_1^2 \beta_j^2] - \frac{1}{2(100^2)\beta_j^2} \right\} \\
& \propto \exp \left\{ -\frac{1}{2} \left[-2 \left(\sum_{t=2}^n \frac{(x_{jt} - \phi_j x_{j,t-1} - (1-\phi_j)\alpha_j)(1-\phi_j)y_t}{(1-\phi_j^2)\sigma_j^2} + \frac{(x_{j1} - \alpha_j)y_1}{\sigma_j^2} \right) \beta_j \right. \right. \\
& \quad \left. \left. + \left(\frac{(1-\phi_j)^2 y_t^2}{(1-\phi_j^2)} + \frac{y_1}{\sigma_j^2} + \frac{1}{100^2} \right) \beta_j^2 \right] \right\} \\
& \implies \beta_j | \dots \sim N(MV^{-1}, V^{-1})
\end{aligned}$$

where

$$\begin{aligned}
M &= \sum_{t=2}^n \frac{(x_{jt} - \phi_j x_{j,t-1} - (1-\phi_j)\alpha_j)(1-\phi_j)y_t}{(1-\phi_j^2)\sigma_j^2} + \frac{(x_{j1} - \alpha_j)y_1}{\sigma_j^2} \\
V &= \frac{(1-\phi_j)^2 y_t^2}{(1-\phi_j^2)} + \frac{y_1}{\sigma_j^2} + \frac{1}{100^2}
\end{aligned}$$

7.5 μ_0 full conditional

$$\begin{aligned}
p(\mu_0 | \dots) & \propto \prod_{t=2}^n p(Y_t | Y_{t-1}, \mu_0) \cdot p(Y_1) \pi(\mu_0) \\
& \propto \exp \left\{ \sum_{t=2}^n -\frac{1}{2(1-\phi_0^2)\sigma_0^2} [y_t - \phi_0 y_{t-1} - (1-\phi_0)\mu_0]^2 - \frac{1}{2\sigma_0^2} [y_1 - \mu_0]^2 - \frac{1}{2(100^2)\mu_0^2} \right\} \\
& \propto \exp \left\{ -\frac{1}{2(1-\phi_0^2)\sigma_0^2} \sum_{t=2}^n [-2(y_t - \phi_0 y_{t-1})(1-\phi_0)\mu_0 + (1-\phi_0)^2 \mu_0^2] \right. \\
& \quad \left. - \frac{1}{2\sigma_0^2} [-2y_1 \mu_0 + \mu_0^2] - \frac{1}{2(100^2)\mu_0^2} \right\} \\
& \propto \exp \left\{ -\frac{1}{2} \left[-2 \left(\sum_{t=2}^n \frac{(y_t - \phi_0 y_{t-1})(1-\phi_0)}{(1-\phi_0^2)\sigma_0^2} + \frac{y_1}{\sigma_0^2} \right) \mu_0 \right. \right. \\
& \quad \left. \left. + \left(\frac{(1-\phi_0)^2}{(1-\phi_0^2)\sigma_0^2} + \frac{1}{\sigma_0^2} + \frac{1}{100^2} \right) \mu_0^2 \right] \right\} \\
& \implies \mu_0 \sim N(MV^{-1}, V^{-1})
\end{aligned}$$

where

$$M = \frac{\sum_{t=2}^n (y_t - \phi_0 y_{t-1})(1 - \phi_0)}{(1 - \phi_0^2)\sigma_0^2} + \frac{y_1}{\sigma_0^2}$$

$$V = \frac{(n-1)(1 - \phi_0)^2}{(1 - \phi_0^2)\sigma_0^2} + \frac{1}{\sigma_0^2} + \frac{1}{100^2}$$

7.6 α_j full conditional

$$p(\alpha_j | \dots) \propto \prod_{t=2}^n p(X_{jt} | X_{j,t-1}, \alpha_j) \cdot p(X_{j1} | \alpha_j) \pi(\alpha_j)$$

$$\propto \exp \left\{ \sum_{t=2}^n -\frac{1}{2(1 - \phi_j^2)\sigma_j^2} [x_{jt} - \phi_j x_{j,t-1} - (1 - \phi_j)\alpha_j - (1 - \phi_j)\beta_j y_t]^2 \right\}$$

$$\cdot \exp \left\{ -\frac{1}{2\sigma_j^2} [x_{j1} - \alpha_j - \beta_j y_1]^2 - \frac{1}{2(100^2)} \phi_j^2 \right\}$$

$$\propto \exp \left\{ \sum_{t=2}^n -\frac{1}{2(1 - \phi_j^2)\sigma_j^2} [-2(x_{jt} - \phi_j x_{j,t-1} - (1 - \phi_j)\beta_j y_t)(1 - \phi_j)\alpha_j + (1 - \phi_j)^2 \alpha_j^2] \right.$$

$$\left. - \frac{1}{2\sigma_j^2} [-2(x_{j1} - \beta_j y_1)\alpha_j + \alpha_j^2] - \frac{1}{2(100^2)} \alpha_j^2 \right\}$$

$$\propto \exp \left\{ -\frac{1}{2} \left[-2 \left(\sum_{t=2}^n \frac{(x_{jt} - \phi_j x_{j,t-1} - (1 - \phi_j)\beta_j y_t)(1 - \phi_j)}{(1 - \phi_j^2)\sigma_j^2} + \frac{x_{j1} - \beta_j y_1}{\sigma_j^2} \right) \alpha_j \right. \right.$$

$$\left. \left. + \left(\frac{(n-1)(1 - \phi_j)^2}{(1 - \phi_j^2)\sigma_j^2} + \frac{1}{\sigma_j^2} + \frac{1}{100^2} \right) \alpha_j^2 \right] \right\}$$

$$\implies \alpha_j | \dots \sim N(MV^{-1}, V^{-1})$$

where

$$M = \sum_{t=2}^n \frac{(x_{jt} - \phi_j x_{j,t-1} - (1 - \phi_j)\beta_j y_t)(1 - \phi_j)}{(1 - \phi_j^2)\sigma_j^2} + \frac{x_{j1} - \beta_j y_1}{\sigma_j^2}$$

$$V = \frac{(n-1)(1 - \phi_j)^2}{(1 - \phi_j^2)\sigma_j^2} + \frac{1}{\sigma_j^2} + \frac{1}{100^2}$$

7.7 σ_0^2 full conditional

$$p(\sigma_0^2 | \dots) \propto \prod_{t=2}^n p(Y_t | Y_{t-1}, \sigma_0^2) \cdot p(Y_1 | \sigma_0^2) \pi(\sigma_0^2)$$

$$\propto (\sigma_0^2)^{-(n-1)/2} \exp \left\{ \frac{1}{\sigma_0^2} \sum_{t=2}^n -\frac{1}{2(1 - \phi_0^2)} [y_t - \phi_0 y_{t-1} - (1 - \phi_0)\mu_0]^2 \right\}$$

$$\cdot (\sigma_0^2)^{-1/2} \exp \left\{ -\frac{1}{2\sigma_0^2} [y_1 - \mu_0]^2 \right\} (\sigma_0^2)^{-0.1-1} \exp \left\{ -\frac{0.1}{\sigma_0^2} \right\}$$

$$\propto (\sigma_0^2)^{-(n/2+0.1)-1} \exp \left\{ -\frac{1}{\sigma_0^2} \left(\sum_{t=2}^n -\frac{1}{2(1-\phi_0^2)} [y_t - \phi_0 y_{t-1} - (1-\phi_0)\mu_0]^2 + \frac{1}{2} [y_1 - \mu_0]^2 + 0.1 \right) \right\}$$

$$\sigma_0^2 \cdots \sim IG \left(n/2 + 0.1, \left(\sum_{t=2}^n -\frac{1}{2(1-\phi_0^2)} [y_t - \phi_0 y_{t-1} - (1-\phi_0)\mu_0]^2 + \frac{1}{2} [y_1 - \mu_0]^2 + 0.1 \right) \right)$$

7.8 σ_j^2 full conditional

$$p(\sigma_j^2 | \dots) \propto \prod_{t=2}^n p(X_{jt} | X_{j,t-1}, \sigma_j^2) \cdot p(X_{j1} | \sigma_j^2) \pi(\sigma_j^2)$$

$$\propto (\sigma_j^2)^{-n/2} \exp \left\{ -\frac{1}{\sigma_j^2} \left(\sum_{t=2}^n -\frac{1}{2(1-\phi_j^2)} [X_{jt} - \phi_j X_{j,t-1} - (1-\phi_j)\mu_{jt}]^2 + \frac{1}{2} [X_{j1} - \mu_{jt}]^2 \right) \right\}$$

$$\cdot (\sigma_j^2)^{-0.1-1} \exp \left\{ -\frac{0.1}{\sigma_j^2} \right\}$$

$$\propto (\sigma_j^2)^{-n/2-0.1-1} \exp \left\{ -\frac{1}{\sigma_j^2} \left(\sum_{t=2}^n -\frac{1}{2(1-\phi_j^2)} [X_{jt} - \phi_j X_{j,t-1} - (1-\phi_j)\mu_{jt}]^2 + \frac{1}{2} [X_{j1} - \mu_{jt}]^2 + 0.1 \right) \right\}$$

$$\sigma_j^2 \cdots \sim IG \left(n/2 + 0.1, \sum_{t=2}^n -\frac{1}{2(1-\phi_j^2)} [X_{jt} - \phi_j X_{j,t-1} - (1-\phi_j)\mu_{jt}]^2 + \frac{1}{2} [X_{j1} - \mu_{jt}]^2 + 0.1 \right)$$

8 Appendix B: Code

```

### Update functions
# update y
yt_impute=function(t, y, X, params){
  xt=X[t,]
  xtp=X[t-1,]
  m1=sum((xt-t*(params$phi_j)*xtp-(1-params$phi_j)*params$alpha_j)
        *(1-params$phi_j)*params$beta_j/
        ((1-params$phi_j^2)*params$sig2_j))
  m2=((y[t+1]-(1-params$phi_0)*params$mu0)*params$phi_0+
        (params$mu0+params$phi_0*(y[t-1]-params$mu0)))/
        ((1-params$phi_0^2)*params$sig2_0)
  M=m1+m2

  V=sum(((1-params$phi_j)^2*params$beta_j^2)/
        ((1-params$phi_j^2)*params$sig2_j))+

```

$$(1+\text{params}\$phi_0^2)/((1-\text{params}\$phi_0^2)*\text{params}\$sig2_0)$$

```

return(rnorm(n=1,mean=M*V^(-1),sd=sqrt(V^(-1))))
}

# update xj
xjt_impute=function(j,t,y,X,params){
  xjtp=X[t-1,j]
  yt=y[t]
  mujt=params$alpha_j[j]+params$beta_j[j]*yt
  if(t==length(y)){
    M=(mujt+params$phi_j[j]*(xjtp-mujt))/
      ((1-params$phi_j[j]^2)*params$sig2_j[j])
    V=1/((1-params$phi_j[j]^2)*params$sig2_j[j])

    return(rnorm(n=1,mean=M*V^(-1),sd=sqrt(V^(-1))))
  } else{

    xjtf=X[t+1,j]
    ytf=y[t+1]
    mujtf=params$alpha_j[j]+params$beta_j[j]*ytf

    M=(mujt+params$phi_j[j]*(xjtp-mujt)+(xjtf-(1-params$phi_j[j])*mujtf)*
      params$phi_j[j])/((1-params$phi_j[j]^2)*params$sig2_j[j])
    V=(1+params$phi_j[j]^2)/((1-params$phi_j[j]^2)*params$sig2_j[j])

    return(rnorm(n=1,mean=M*V^(-1),sd=sqrt(V^(-1))))
  }
}

# update beta_j

```

```

beta_j_update=function(j ,y ,X, params){
  xj=X[, j]
  xj1=xj[1]
  xjrest=xj[-1]

  m1=sum(( xjrest -params$phi_j[j]*xj[1:(length(xj)-1)]-
    (1-params$phi_j[j])*params$alpha_j[j])*(1-params$phi_j[j])*
    y[2:length(y)]/((1-params$phi_j[j]^2)*params$sig2_j[j]))
  m2=(xj1-params$alpha_j[j])*y[1]/params$sig2_j[j]
  M=m1+m2

  V=sum((1-params$phi_j[j])^2*y[2:length(y)]^2/((1-params$phi_j[j]^2)*
    params$sig2_j[j]))+y[1]^2/params$sig2_j[j]+1/(100^2)

  return(rnorm(n=1,mean=M*V^(-1),sd=sqrt(V^(-1))))
}

```

update alpha_j

```

alpha_j_update=function(j ,y ,X, params){
  xj=X[, j]
  xj1=xj[1]
  xjrest=xj[-1]

  m1=sum(( xjrest -params$phi_j[j]*xj[1:(length(xj)-1)]-
    (1-params$phi_j[j])*params$beta_j[j]*
    y[2:length(y)]*(1-params$phi_j[j])/
    ((1-params$phi_j[j]^2)*params$sig2_j[j]))
  m2=(xj1-params$beta_j[j])*y[1]/params$sig2_j[j]
  M=m1+m2

  V=(length(y)-1)*(1-params$phi_j[j])^2/
    ((1-params$phi_j[j]^2)*params$sig2_j[j])+

```

```

1/params$sig2_j[j]+1/(100^2)

return(rnorm(n=1,mean=M*V^(-1),sd=sqrt(V^(-1))))
}

# update mu0
mu0_**update=**function(y, params){
  M=**sum((y[2:length(y)]-params$phi_0*y[1:(length(y)-1)])*
    (1-params$phi_0))/((1-params$phi_0^2)*
    params$sig2_0)+y[1]/params$sig2_0

  V=(length(y)-1)*(1-params$phi_0)^2/
    ((1-params$phi_0^2)*params$sig2_0)+
    1/params$sig2_0+1/(100^2)

  return(rnorm(n=1,mean=M*V^(-1),sd=sqrt(V^(-1))))
}

# update sig2_0
sig2_0_**update=**function(y, params){
  a=length(y)/2+0.1
  b=**sum((y[2:length(y)]-params$phi_0*y[1:(length(y)-1)]
    -(1-params$phi_0)*params$mu0)^2)/
    (2*(1-params$phi_0^2))+
    (y[1]-params$mu0)^2/2+0.1
  return(1/rgamma(n=1,a,b))
}

# update sig2_j
sig2_j_**update=**function(j,y,X,params){
  xj=X[,j]
  muj=params$alpha_j[j]+params$beta_j[j]*y

```

```

a=length(xj)/2+0.1
b=sum((xj[2:length(xj)]-params$phi_j[j]*xj[1:(length(xj)-1)]
      -(1-params$phi_j[j])*
      muj[2:length(xj)]^2)/(2*(1-params$phi_j[j]^2))+
      (xj[1]-muj[1])^2/2+0.1

return(1/rgamma(n=1,a,b))
}

##MH for phi
phi_0_likeli=function(y,params,phi){
  lf=log(dnorm(y[1],mean=params$mu0,sd=sqrt(params$sig2_0)))
  for(t in 2:length(y)){
    lf=lf+log(dnorm(y[t],mean=params$mu0+phi*(y[t-1]-params$mu0),
                    sd=sqrt((1-phi^2)*params$sig2_0)))
  }
  return(lf)
}

phi_j_likeli=function(j,y,X,params,phi){
  lf=log(dnorm(X[1,j],mean=params$alpha_j[j]+params$beta_j[j]*y[1],
                sd=sqrt(params$sig2_j[j])))
  for(t in 2:length(y)){
    lf=lf+log(dnorm(X[t,j],mean=params$alpha_j[j]+params$beta_j[j]*y[t]+
                    phi*(X[t-1,j]-params$alpha_j[j]-params$beta_j[j]*y[t]),
                    sd=sqrt((1-phi^2)*params$sig2_j[j])))
  }
  return(lf)
}

phi_0_update=function(y,params){

```

```

phi=params$phi_0
phi_0_star=rbeta(n=1,0.5,(0.5-0.5*phi)/phi) #proposal distribution

R=phi_0_likeli(y,params,phi_0_star)-phi_0_likeli(y,params,phi)+
log(dbeta(phi,0.5,(0.5-0.5*phi_0_star)/phi_0_star))-
log(dbeta(phi_0_star,0.5,(0.5-0.5*phi)/phi))

if(log(runif(1))<R){phi=phi_0_star}

return(phi)
}

phi_j_update=function(j,y,X,params){
phi=params$phi_j[j]
phi_j_star=rbeta(n=1,0.5,(0.5-0.5*phi)/phi) #proposal distribution

R=phi_j_likeli(j,y,X,params,phi_j_star)-
phi_j_likeli(j,y,X,params,phi)+
log(dbeta(phi,0.5,(0.5-0.5*phi_j_star)/phi_j_star))-
log(dbeta(phi_j_star,0.5,(0.5-0.5*phi)/phi))

if(log(runif(1))<R){phi=phi_j_star}

return(phi)
}

dat=read.csv('E1dat.csv')
dat

### MCMC
y=dat$RFI
y_init=(y-mean(y,na.rm=TRUE))/sd(y,na.rm=TRUE) #center and scale Y
y_init[which(is.na(y))]=0

```



```

X=as.matrix(dat[,c(2:7)])
X_init=X
for(j in 1:6){
  X_init[which(is.na(X[,j])),j]=0 #proxies are centered and scaled
}

y_current=y_init
X_current=X_init

#Initializing parameters
params=list()

params$phi_j=c(0.5,0.5,0.5,0.5,0.5,0.5)
params$beta_j=rep(0,6)
params$alpha_j=rep(0,6)
params$sig2_j=rep(1,6)
params$phi_0=0.05
params$sig2_0=1
params$mu0=mean(y_init)

#Posterior list
posterior=list()

posterior$y=matrix(y_init)

posterior$X1=matrix(X_init[,1])
posterior$X2=matrix(X_init[,2])
posterior$X3=matrix(X_init[,3])
posterior$X4=matrix(X_init[,4])

```

```

posterior$X5=matrix(X_init[,5])
posterior$X6=matrix(X_init[,6])

posterior$phi_j=matrix(params$phi_j)
posterior$beta_j=matrix(params$beta_j)
posterior$alpha_j=matrix(params$alpha_j)
posterior$sig2_j=matrix(params$sig2_j)
posterior$phi_0=params$phi_0
posterior$sig2_0=params$sig2_0
posterior$mu0=params$mu0

#MCMC
burn=2000
iters=20000

rfi_missing=which(is.na(y))
keep_0=0
keep_j=rep(0,6)
for(i in 1:iters){
  if(i%1000==0){print(i)}
  #RFI imputation
  y_update=y_current
  for(t in rfi_missing){
    if(t==1){
      y_update[t]=rnorm(n=1,mean=params$mu0,sd=sqrt(params$sig2_0))
    } else{
      y_update[t]=yt_impute(t,y_current,X_current,params)
    }
  }
  y_current=y_update
  posterior$y=cbind(posterior$y,y_current)
}

```

```

#Proxy imputation
X_update=X_current
for(j in 1:6){
  proxy_missing=which(is.na(X[,j]))
  for(t in proxy_missing){
    if(t==1){
      X_update[t,j]=rnorm(n=1,params$alpha_j[j]+params$beta_j[j]*
        y_current[1],sqrt(params$sig2_j[j]))
    } else{
      X_update[t,j]=xjt_impute(j,t,y_current,X_current,params)
    }
  }
}
X_current=X_update

posterior$X1=cbind(posterior$X1,X_current[,1])
posterior$X2=cbind(posterior$X2,X_current[,2])
posterior$X3=cbind(posterior$X3,X_current[,3])
posterior$X4=cbind(posterior$X4,X_current[,4])
posterior$X5=cbind(posterior$X5,X_current[,5])
posterior$X6=cbind(posterior$X6,X_current[,6])

#update alphas.....
alpha_update=params$alpha_j
for(j in 1:6){
  #alpha_update[j]=alpha_j_update(j,y_current,X_current,params)
}
params$alpha_j=alpha_update
posterior$alpha_j=cbind(posterior$alpha_j,alpha_update)

#update betas
beta_update=params$beta_j

```

```

for (j in 1:6){
  beta_update[j]=beta_j_update(j ,y_current ,X_current ,params)
}
params$beta_j=beta_update
posterior$beta_j=cbind( posterior$beta_j ,beta_update)

#update sig2_0
sig20_update=sig2_0_update(y_current ,params)
params$sig2_0=sig20_update
posterior$sig2_0=cbind( posterior$sig2_0 ,sig20_update)

#update sig2_j
sig2j_update=params$sig2_j
for (j in 1:6){
  sig2j_update[j]=sig2_j_update(j ,y_current ,X_current ,params)
}
params$sig2_j=sig2j_update
posterior$sig2_j=cbind( posterior$sig2_j , sig2j_update)

#update mu0
mu0_current=mu0_update(y_current ,params)
params$mu0=mu0_current
posterior$mu0=cbind( posterior$mu0 ,mu0_current)

#update phi_0
phi_0_current=params$phi_0
phi_0_current=phi_0_update(y_current ,params)
if (phi_0_current !=params$phi_0){ keep_0=keep_0+1}
params$phi_0=phi_0_current
posterior$phi_0=cbind( posterior$phi_0 ,phi_0_current)

#update phi_j

```

```

phi_j_current=params$phi_j
for(j in 1:6){
  phi_j_current[j]=phi_j_update(j,y_current,X_current,params)
  if(phi_j_current[j]!=params$phi_j[j]){keep_j[j]=keep_j[j]+1}
}
params$phi_j=phi_j_current
posterior$phi_j=cbind(posterior$phi_j,phi_j_current)
}

```

#Diagnostics and computation

```

par(mfrow=c(1,3))
plot((burn:iters),posterior$phi_0[1,(burn:iters)],type="l")
plot((burn:iters),posterior$sig2_0[1,(burn:iters)],type="l")
plot((burn:iters),posterior$mu0[1,(burn:iters)],type="l")

```

```

par(mfrow=c(2,3))
for(j in 1:6){
  plot((1:iters),posterior$phi_j[j,(1:iters)],type="l")
}

```

```

for(j in 1:6){
  acf(posterior$phi_j[j,])
}

```

```

for(j in 1:6){
  plot((burn:iters),posterior$beta_j[j,(burn:iters)],
  type="l",xlab="Iterations",ylab="")
  title(paste0("beta",j))
}

```

```

for(j in 1:6){
  acf(posterior$beta_j[j,])
}

#Effective sample size calculations and Geweke test

for(j in 1:6){
  ess=iters/(1+2*sum(acf(posterior$phi_j[j,], plot=FALSE)[[1]][1:5]))
  print(ess)

  t=mcmc(posterior$phi_j[j,])
  print(geweke.diag(t)[1])
}

for(j in 1:6){
  ess=iters/(1+2*sum(acf(posterior$beta_j[j,], plot=FALSE)[[1]]))
  print(ess)

  t=mcmc(posterior$beta_j[j,])
  print(geweke.diag(t)[1])
}

for(j in 1:6){
  ess=iters/(1+2*sum(acf(posterior$sig2_j[j,], plot=FALSE)[[1]]))
  print(ess)

  t=mcmc(posterior$sig2_j[j,])
  print(geweke.diag(t)[1])
}

iters/(1+2*sum(acf(posterior$phi_0[1,], plot=FALSE)[[1]]))
t=mcmc(posterior$phi_0[1,])

```

```

print(geweke.diag(t)[1])

iters/(1+2*sum(acf(posterior$sig2_0[1,], plot=FALSE)[[1]]))
t=mcmc(posterior$sig2_0[1,])
print(geweke.diag(t)[1])

iters/(1+2*sum(acf(posterior$mu0[1,], plot=FALSE)[[1]]))
t=mcmc(posterior$mu0[1,])
geweke.diag(t)[1]

#RFI prediction
ymeans=NULL
y95=NULL
y05=NULL
for(i in 1:length(y)){
  ymeans=c(ymeans,mean(posterior$y[i,(burn:iters)]))
  y95=c(y95,quantile(posterior$y[i,(burn:iters)],0.975))
  y05=c(y05,quantile(posterior$y[i,(burn:iters)],0.025))
}

y_fit=ymeans*sd(y,na.rm=TRUE)+mean(y,na.rm=TRUE)
y_fit95=y95*sd(y,na.rm=TRUE)+mean(y,na.rm=TRUE)
y_fit05=y05*sd(y,na.rm=TRUE)+mean(y,na.rm=TRUE)

ggplot(mapping=aes(x=dat$Year[239:339],
y=y_fit[239:339]),xlab="Year",ylab="RFI") +
  geom_line() +
geom_ribbon(aes(ymin=y_fit05[239:339],ymax=y_fit95[239:339],
fill="95% credible interval"),alpha=0.5,linetype="dotted")+
  ggtitle('Reconstructed RFI back to 1850 with uncertainty') +
  labs(y='RFI',x='Year') +

```

```

scale_fill_manual("", values="grey12") +
theme_bw()

keep_0/iters
keep_j/iters

## Testing
beta_j_CI=matrix(nrow=6,ncol=3)
for(j in 1:6){
  beta_j_CI[j,1]=quantile(posterior$beta_j[j,],0.025)
  beta_j_CI[j,2]=mean(posterior$beta_j[j,])
  beta_j_CI[j,3]=quantile(posterior$beta_j[j,],0.975)
}
colnames(beta_j_CI)=c("q=.05", "q=0.5", "q=0.95")
beta_j_CI

par(mfrow=c(2,3))
for(j in 1:6){
  hist(posterior$beta_j[j,])
}

## RFI Trend
start=c(1612,1600+(1:7)*50,2000)
end=c(1649,1650+(1:7)*50-1,2017)

y_trends=matrix(0,nrow=9,ncol=iters)
for(i in 1:iters){

```



```

for (k in 1:9){
  y_trends[k,i]=y_trends[k,i]+
  mean(posterior$y[(start[k]-1611):(end[k]-1611),i])
}
}

y_trends_ mean=matrix(0,nrow=9,ncol=3)
for (k in 1:9){
  y_trends_ mean[k,1]=quantile(y_trends[k,],0.025)
  y_trends_ mean[k,2]=mean(y_trends[k,])
  y_trends_ mean[k,3]=quantile(y_trends[k,],0.975)
}

y_trends_fit=y_trends_ mean*sd(y,na.rm=TRUE)+mean(y,na.rm=TRUE)
ggplot(mapping=aes(x=start,y=y_trends_fit[,2]),xlab="Year",ylab="RFI") +
geom_line() +
geom_ribbon(aes(ymin=y_trends_fit[,1],ymax=y_trends_fit[,3],
fill="95% credible interval"),alpha=0.3,linetype="dotted")+
ggtitle('50-year mean RFI') +
labs(y='RFI',x='Year')+
scale_fill_manual("",values="grey12")+
theme_bw()

cbind(y_trends_fit[c(1:8),3],rep(y_trends_fit[9,1],8))

##### #Simulated data
phi_0=0.1
phi_j=rep(0.2,6)
alpha_j=c(2,4,3,1,-7,0)
beta_j=c(0.1,0.1,0.5,0.1,0.9,0.05)
sig2_0=1
sig2_j=rep(1,6)
mu0=100

```

```

y=rnorm(n=1,mu0,sd=sqrt(sig2_0))
for(i in 1:99){
  yt=rnorm(n=1,mu0-phi_0*(y[i]-mu0),sd=sqrt((1-phi_0^2)*sig2_0))
  y=c(y,yt)
}

X=matrix(NA,nrow=100,ncol=6)
for(j in 1:6){
  X[1,j]=rnorm(n=1,alpha_j[j]+beta_j[j]*y[1],sd=sqrt(sig2_j[j]))
  for(i in 2:100){
    X[i,j]=rnorm(n=1,alpha_j[j]+beta_j[j]*y[i]+phi_j[j]*(X[(i-1),j]-
      alpha_j[j]-beta_j[j]*y[i]),sd=sqrt((1-phi_j[j]^2)*sig2_j[j]))
  }
}

nas=sample(1:99,30)
y.dat=y
y.dat[c(1:50)]=NA
y_init=y.dat
y_init=(y_init-mean(y_init,na.rm=TRUE))
/sd(y_init,na.rm=TRUE) #center and scale y
y_init[which(is.na(y.dat))]=0
#y_init[which(is.na(y.dat))]=mean(y_init,na.rm=TRUE)

X.dat=X
for(j in 1:6){
  nas=sample(1:99,30)
  X.dat[nas,j]=NA
}

```

```
X_init=X.dat
for(j in 1:6){
  X_init=(X_init - mean(X_init[,j], na.rm=TRUE))/sd(X_init[,j], na.rm=TRUE)
  X_init[which(is.na(X.dat[,j])),j]=0#proxies are centered and scaled
  #X_init[which(is.na(X.dat[,j])),j]=mean(X_init[,j], na.rm=TRUE)
}

y_current=y_init
X_current=X_init
```