ST740 – Assignment 2 – Due 9/21

This homework is due in class on Wednesday, 9/21. Show work for questions that say "derive" and include code used to produce the results. You may work in groups, but write up separate solutions.

1. Assume $Y_1, ..., Y_n|\theta \sim \text{Normal}(0, \theta)$ and variance parameter has prior $\theta \sim \text{Gamma}(a, b)$.
   (a) Derive the posterior distribution of θ, i.e., give a parametric family like $\theta|Y \sim \text{Beta}(Y, a^b)$.[1]
   (b) Would you say this prior is conjugate? Justify your answer.

2. Say $\mathbf{Y} = (Y_1, ..., Y_p)|\theta \sim \text{Multinomial}(n; \theta)$ for $\theta = (\theta_1, ..., \theta_p)$ so that the likelihood is

$$f(Y|\theta) = \frac{n!}{\prod_{j=1}^{p} Y_j!} \prod_{j=1}^{p} \theta_j^{Y_j}.$$

   (a) Derive the Jeffreys prior for θ.
   (b) Derive the posterior under the prior in (a).
   (c) Assume that $\mathbf{Y} = (10, 20, 30)$ and summarize the posterior under the prior in (a) in a figure and table.
   (d) Now apply the Bayesian Central Limit Theorem to obtain an approximate normal distribution for the posterior of θ given $\mathbf{Y} = (10, 20, 30)$. Summarize this approximate posterior in a figure and table. Are the results similar to the exact posterior? Is this a good approximation?

3. Assume $Y_i|\theta \sim \text{Uniform}(0, \theta)$ independent for $i \in \{1, ..., n\}$ and prior $\theta \sim \text{Pareto}(\theta_0, \alpha)$ with support $\theta > \theta_0$ and CDF $\text{Prob}(\theta < t) = 1 - (\theta_0/t)^\alpha$.
   (a) Say the true value of θ is $\theta^* = 10$ and the prior has $\theta_0 = \alpha = 1$. For a dataset of size n, $\mathbf{Y}_n = \{Y_1, ..., Y_n\}$, let $p_n = \text{E}_{\mathbf{Y}_n|\theta^*}\{\text{Prob}(\theta^* - \epsilon < \theta < \theta^* + \epsilon|\mathbf{Y}_n)\}$ for $\epsilon = 0.1$. Compute a Monte Carlo approximation to p_n for each $n \in \{10, 20, ..., 1000\}$. Does a plot of n versus p_n suggest posterior consistency? Why?
   (b) Without evoking any general theorems discussed in class, derive $\lim_{n\to\infty} p_n$. Do you get the same conclusion about posterior consistency as the Monte Carlo study in (a)?

[1] The posterior family is fairly obscure.